Can optimism be a remedy for present bias?*

Minwook Kang  Lei Sandy Ye

July, 2020

Abstract

Under economies with hyperbolic preferences, vast research has investigated welfare-improving tax policies to resolve capital misallocation issues. In this paper, we suggest an alternative channel to overcome a form of this issue associated with consumer’s present bias – optimism, as defined by overexpectation of future productivity. We show that even though optimism negatively impacts consumers under normal circumstances, a moderate level of it can be beneficial when consumers have hyperbolic preferences. On the other hand, pessimism always negatively impacts consumer welfare. A steady-state analysis shows that the quantitative impact of optimism on welfare can be sizable.

Keywords: Optimism; Present bias; Hyperbolic discounting; Undersavings; Pessimism

*Minwook Kang is at the Division of Economics, School of Social Sciences, Nanyang Technological University (E-mail: mwkang@ntu.edu.sg) and Lei Sandy Ye is at the World Bank (E-mail: lye1@worldbank.org). An earlier version of this paper circulated under the title of “Optism, Present Bias, and the Macroeconomy.” We appreciate helpful comments from two anonymous referees and the participants of the AEI-Five Joint Conference in 2017 at Seoul. The findings, interpretations, and conclusions expressed in this paper are entirely those of the authors. They do not necessarily represent the views of the World Bank, its Executive Directors, or the countries they represent. Kang (Corresponding author) acknowledges the research support from Nanyang Technological University (NTU) AcRF Tier-1 Grant (RG62/18). All errors are our own.
1. Introduction

Extensive introspective and empirical evidence suggest that consumers’ discounting functions are approximately hyperbolic rather than exponential (Ainslie 1992; Loewenstein and Prelec 1992; Angeletos et al. 2001). Based on such evidence, Strotz (1956), Phelps and Pollak (1968), and Laibson (1996, 1997) developed a time-inconsistent quasi-hyperbolic discounting model. With time-inconsistent preferences, a sequence of selves plays a dynamic game, which results in suboptimal equilibria. Therefore, governments have incentives to use various types of tax policies to curb consumers’ myopic decisions. Under macroeconomic frameworks, consumption tax (Laibson 1997; O’Donoghue and Rabin 2006), savings subsidy (Krusell, Kuru¸sçu, and Smith 2010, Pavoni and Yazici 2017), and income tax (Kang 2019) have been proposed as welfare-improving tax policies under settings of time-inconsistent consumers. This paper introduces an alternative way to improve welfare without the use of tax policies. All of the tax policies aforementioned are revenue-neutral policies – if a subsidy exists, there should be a tax to make the government’s revenue unaffected. These policies incur administrative costs and thus, potentially consumer resistance. On the other hand, this paper proposes producer optimism, which may arise from less costly government actions, as a potential remedy for consumers’ present bias.

Research evidence suggests that the indirect impact on optimism associated with government actions (e.g., presidential speech, new policy announcements) affects the producer’s expectation of future productivity. Although these actions are not tax policies, this paper shows that under a hyperbolic economy, they have a similar impact.

Maintaining positive sentiments in the corporate sector has long been an important objective of policymakers. A number of studies in the academic literature have shown that perceptions on economic activity drives economic outcomes. Theoretically, overoptimism may help situate the economy on a new equilibrium path under environments of strong complementarity, such as in Cooper and John (1988). Rodriguez Mora and Schulstad (2007) show for European economies that GNP announcements that deviate from the actual can impact investment and economic activity. Similar

---


2In our model, optimism is defined by the representative firm as having higher expectation about its future total factor productivity (TFP) than actual productivity. This overestimation of productivity can be attributed to producers’ overconfidence (see Malmendier and Tate 2005) or optimism (see Campbell et al. 2011). An overconfident producer is one who overestimates its own ability in pursuing a project, while an optimistic producer is one who overestimates the project’s return due to optimism about the macroeconomic environment. Since both overconfidence and optimism lead to overestimation of the project return, our model can be interpreted to reflect both types of firms.
lar evidence are shown by Blanchard, Lorenzoni, and L’Huillier (2017) and Enders, Kleemann, and Mueller (2017). For example, in monetary policy, policymakers can consider its growth forecasts based on a slightly more optimistic path of expected future corporate revenue by operating on a baseline where corporate managers invest in all long-term NPV projects, thereby helping to manage expectations (King, Lu, and Pasten 2008). In fiscal policy, policymakers can consider budget forecasts based on a baseline of low political stability and risk, factors that support investment activity (Julio and Yook 2012; Gulen and Ion 2012).

During economic recessions, government policy announcement and communication that strongly signal reduced policy uncertainty in the future could be warranted as well, as evidence has shown that positive sentiments are especially important during these times (Bachmann and Sims 2012). These actions could be exemplified by president’s speeches to boost confidence in economic outlook or signal stability in future government regulations for the private sector, or statements by monetary policymakers to anchor market confidence.

The notion of optimism has also been featured prominently in economics research. It is widely believed that optimism or self-confidence can affect not only individuals’ choices but also macroeconomic variables. As early as 1936, Keynes has argued that optimism can impact macroeconomic activities beyond what can be explained by fundamentals.\(^3\) Recently, theoretical and empirical studies of optimism have been more active at the micro-level, particularly in corporate finance. Optimism has been found to be a prevalent feature of corporate managers, resulting from their cognitive bias. In the most basic setting, if corporate managers overforecast the productivity or value of their firms, investment would be higher relative to the rationally-optimal level, leading to overinvestment. In the corporate finance literature, evidence shows that CEOs frequently demonstrate excessive optimism.\(^4\) Even though we have emphasized the normative aspect of producer optimism as a possible policy tool in this paper, a related positive point is also relevant – producer optimism can be beneficial in the hyperbolic economy. The finance literature generally considers optimism to deteriorate firm value and shareholder’s welfare (see Hayward and Hambrick, 1997; Malmendier and Tate, 2008; John, Liu, and Taffler, 2010). However, this paper shows that over-optimism in some instances can be counterintuitively beneficial in an economy with

\(^3\)In his classic, The General Theory of Employment, Interest and Money, Keynes remarked that “Even apart from the instability due to speculation, there is the instability due to the characteristic of human nature that a large proportion of our positive activities depends on spontaneous optimism rather than mathematical expectation, whether moral or hedonistic or economic.”

\(^4\)The literature on managerial optimism in corporate finance is vast. For examples of these works, see Malmendier and Tate (2005, 2008, 2015); Otto (2014); Hayward and Hambrick (1997); Doukas and Petmezas (2007); Billett and Qian (2008); John, Liu, and Taffler (2010); Campbell et al. (2011); Ben-David, Graham, and Harvey (2013); Banerjee et al. (2015). This literature also shows that even though an overly-optimistic CEO acts in a way that negatively impacts firm value, there remain incentives for firms to hire them (Hirshleifer, Low, and Hong 2012; Goel and Thakor 2008; Galasso and Simcoe 2011).
present-biased consumers.

Nevertheless, most macroeconomic models assume that the representative producer perfectly forecasts its future productivity. However, this departs from reality, given the inherent unpredictability of government policies, oil prices, political stability, among other developments. Therefore, the following assumptions are more reasonable: (1) the producers’ expectations of future productivity necessarily contain errors and (2) government actions can influence these expectations. Under these assumptions, a natural question is what is the impact of producer optimism/pessimism on the economy. Producers’ optimism, as defined by excessive expectation of future productivity, will increase the real interest rate in the capital market. However, optimism would decrease dividend payouts via lower firm profits. Thus, the consumer will save more due to higher interest rates but her dividend income will be lower. In this paper, we show that if the consumer’s preference is hyperbolically present-biased, the increased savings income can outweigh the decreased dividend income, so under sustained optimism, the consumer’s total income can increase despite a decrease in producer profit. Therefore, even though optimism clearly negatively impacts producers, it can positively impact hyperbolic consumers’ welfare. We also show that pessimism negatively affects consumers’ welfare.

The main difference between hyperbolic discounting and exponential discounting is that in the former, future producer’s optimism can positively affect current consumer’s intertemporal utility. Future optimism will increase future real interest rate, which in turn increases future savings. Under hyperbolic discounting, the additional increase in future savings can increase current intertemporal utility. This is because the hyperbolic consumer suffers from future self-control problems, which induces undersaving in the future. Optimism can be a remedy for the undersaving problem by increasing current intertemporal utility. On the other hand, under exponential discounting, where there is no self-control problem, optimism will certainly impact current intertemporal utility negatively.

In sum, current period’s optimism increases capital tomorrow but decreases welfare in the current period. However, future optimism can improve welfare in the current period. Therefore, this paper shows that the joint existence of optimism in both periods, that is, sustained optimism, can lead to Pareto-improvement of consumption. Under the same logic, pessimism can lead to Pareto-disimprovement.

We also extend our framework to an infinite period model and perform a steady-state analysis. This analysis helps quantify the potential role of optimism in the macroeconomy. Our steady-state analysis shows that optimism, in the form of over-

---

5 Among research papers that adopt the infinite-period model to assess quantitative implications of present bias. Laibson (1997) quantitatively investigated the welfare loss when the economy with hyperbolic consumers experiences financial innovation, which deprives consumers of illiquid asset savings. Angeletos et al. (2001) compared the macroeconomic model with and without present bias, and showed that the present-biased model explains the macroeconomic variables better than the standard exponential discounting model. Nakajima (2012) quantitatively analyzed the welfare loss from relaxing the borrowing constraint in a macroeconomic model with temptation utility.
estimation of productivity by 7 percent, can raise the equilibrium capital-to-output ratio from 2.8 to 3, where 2.8 corresponds to the capital-to-output ratio associated with the hyperbolic discounting parameter (β) of 0.7 and no optimism. On average, a one percentage point degree of optimism on future total factor productivity (TFP) can increase the steady-state capital-to-output ratio by about one percentage point.

An important question the infinite period analysis raises is that once sustained optimism enters the economy, can optimism increase consumers’ intertemporal utilities, especially that of the initial consumer? Different from future consumers, who would benefit from the additional capital accumulation associated with sustained optimism, the initial consumer cannot directly benefit. Therefore, the welfare gain for the initial consumer’s intertemporal utility is the lowest among all periods’ intertemporal utilities. Nevertheless, this paper shows that the welfare gain associated with a 7-percentage-point increase in optimism is equivalent to about 4-percentage-point additional consumption good for the initial consumer. The welfare gain for the future consumer in the new steady state can be as large as 43 percent additional consumption good in the period considered. These quantitative welfare gains suggest a potential role for government’s efforts to boost producers’ optimism.

Most literature assumes that overconfidence and optimism are exogenously given, which negatively impacts firm value. On the other hand, as shown in Bénabou and Tirole (2002), overconfidence can be endogenously generated to help curb myopic decisions by an individual with hyperbolically-discounted time preferences. They show that the current-self might intentionally choose self-deception, which could provide the future-self inaccurate information (overconfident motivation) about her ability to make a project succeed. This self-deception increases the likelihood of the future-self undertaking the project, which would positively impact the current self’s welfare. Even though this paper assumes that optimism is exogenous for the producer, the main result of this paper can allow for endogenously-generated optimism if one assumes that optimism level can be affected by a benevolent government.

The remainder of the paper is organized as follows. In Section 2, we introduce a three-period model incorporating producers’ optimism and consumers’ present bias. Section 3 investigates the impact of one-time optimism on producers’ profits and consumers’ welfare. Section 4 investigates the impact of sustained optimism and pessimism on consumers’ welfare. In Section 5, we extend our framework to an infinite-period model and present quantitative results. Section 6 concludes. All proofs are in the appendices.

2. Three-period model

In this section, we introduce a three-period macroeconomic model in which the representative consumer has present-biased preferences and the representative firm has irrationally optimistic views on its productivity. To incorporate time-inconsistency
into the consumption-savings model, we need at least two distinct decision periods. Because there is no decision-making in the last period, at least three periods are necessary to incorporate hyperbolic-discounted time preferences to our macroeconomic model. This three-period model sheds light on the basic mechanism behind how present bias and optimism can jointly improve welfare, despite negatively impacting welfare when considered separately. We later extend this three-period framework to an infinite-period model in Section 5 to quantify the impact of producers’ optimism on the hyperbolic consumer’s consumption-saving decisions and welfare.

2.1. A representative consumer with hyperbolic discounting

There is a representative agent who lives three periods, \( t = 0, 1, 2 \). The consumer takes on the period utility, \( u(c) \), which is strictly increasing, strictly concave, and twice continuously differentiable. It also satisfies the limiting condition, \( \lim_{c \to 0} u'(c) = \infty \), where \( c \) is a perishable consumption good. The representative consumer is endowed with one unit of labor good, which has an inelastic supply. The consumer is endowed with \( k_0 \) units of capital in period 0. The capital depreciates at a rate of \( d \in (0, 1) \).

The consumer’s resource constraints are \( c_0 + k_1 = r_0 k_0 + (1 - d) k_0 + w_0 + \pi_0 \) in period 0, \( c_1 + k_2 = r_1 k_1 + (1 - d) k_1 + w_1 + \pi_1 \) in period 1, and \( c_2 = r_2 k_2 + (1 - d) k_2 + w_2 + \pi_2 \) in period 2, where \( r_t, w_t, \) and \( \pi_t \) are the real rental rate of capital, real wage, and the dividend income (firm’s profit), respectively, in period \( t \). With linear homogenous production functions, as assumed in conventional macroeconomic models, the firm’s profit is zero. However, we allow the firm to misestimate its productivity, which could induce the profit to deviate from zero.

The intertemporal utilities in the three periods, \( U^{(0)}, U^{(1)}, \) and \( U^{(2)} \) are

\[
U^{(0)}(c_0, c_1, c_2) = u(c_0) + \beta \left( \delta u(c_1) + \delta^2 u(c_2) \right),
\]

\[
U^{(1)}(c_1, c_2) = u(c_1) + \beta \delta u(c_2),
\]

and

\[
U^{(2)}(c_2) = u(c_2),
\]

where \( \delta \in (0, 1) \) is a long-run discount factor and \( \beta \in (0, 1) \) is a hyperbolic discount factor. If \( \beta = 1 \), the consumer’s preference follows exponential discounting and thus would be a time-consistent decision maker. If \( \beta < 1 \), the consumer is time-inconsistent in the sense that the marginal utility of current consumption is higher than that of a future period, i.e. the marginal utility of \( c_1 \) relative to the marginal utility of \( c_2 \) in term of \( U^{(1)} \) is higher than that in term of \( U^{(0)} \).

The consumer perfectly forecasts future market prices and profits, and also knows her future preferences (sophisticated consumer) in the present period. Therefore, we can solve the consumer’s maximization problems through backward induction. Given \( (r_t, w_t, \pi_t)_{t=1}^2 \), the period-2 consumer consumes all her financial and labor income. Then, the period-1 consumer solves the following maximization problem, conditional
on $k_1$:

$$\max_{k_2|k_1} U^{(1)}(r_1 k_1 + (1 - d) k_1 + w_1 + \pi_1 - k_2, r_2 k_2 + (1 - d) k_2 + w_2 + \pi_2).$$  \hspace{1cm} (1)$$

From the maximization problem of Eq. (1), we implicitly derive $k_2$ as a function of $k_1$, conditional on $(r_t, w_t, \pi_t)_{t=1}^2$, denoted as $\bar{k}_2(k_1)$. Given the savings response function $\bar{k}_2(k_1)$ and $(r_t, w_t, \pi_t)_{t=0}^2$, the consumer chooses $k_1$ to maximize $U^{(0)}$:

$$\max_{k_1} U^{(0)}\left(\begin{array}{l}
r_0 k_0 + (1 - d) k_0 + w_0 + \pi_0 - k_1, \\
r_1 k_1 + (1 - d) k_1 + w_1 + \pi_1 - \bar{k}_2(k_1), \\
r_2 \bar{k}_2(k_1) + (1 - d) \bar{k}_2(k_1) + w_2 + \pi_2 \end{array}\right)$$  \hspace{1cm} (2)$$

The consumer’s optimal choice of capital can be characterized as a subgame perfect Nash equilibrium $(k_1^*, \bar{k}_2(k_1))$ such that $\bar{k}_2(k_1)$ solves the period-1 maximization problem of Eq. (1), conditional on $k_1$; and $k_1^*$ solves the period-0 maximization problem of Eq. (2).

### 2.2. A representative firm with optimistic views

We define a representative firm’s production function in period $t$ as $A_t F(K_t, N_t)$, where $A_t$, $K_t$, and $N_t$ represent the period-$t$ total factor productivity, aggregate capital, and aggregate labor, respectively. The function $F(K_t, N_t)$ satisfies the Inada conditions and exhibits constant returns to scale. We define per-worker production function as $f_t(k_t) = A_t F(K_t, N_t)/N_t$, where $k_t (= K_t/N_t)$ is the representative consumer’s per-capita capital in period $t$.

We introduce the notion of optimism for producers as follows: the firm believes that current investment can yield higher returns than actual return. We parameterize optimism in period $t$ as “$m_t$”, where the firm in period $t - 1$ believes that the period-$t$ productivity is $(1 + m_t)A_t$ instead of $A_t$. This approach to model optimism relies on a large theoretical literature in which the optimistic parameter and real productivity parameter are distinct (see Malmendier and Tate 2005 and Campbell et al. 2011). Thus, $m_t$ being positive (negative) means that the firm has overly optimistic (pessimistic) views about its productivity. The firm chooses the optimal level of period-$t$ capital in period $t - 1$ based on the misestimated production function $(1 + m_t)A_t F(K_t, N_t)$. Therefore, when the firm has optimistic view (i.e. $m_t > 0$), its estimated marginal return of capital is higher than the actual return, which results in higher demand for investment in capital. In period $t - 1$, the firm makes decision on capital investment based on the optimistic productivity $(1 + m_t)A_t$. However, in period T, the market clearing condition is determined by the real productivity $A_t$. Therefore, the misestimation of productivity results in a decrease in profit, which will be introduced in the next subsection.

In period $t - 1$, the firm chooses the period-$t$ capital and labor levels based on the
following maximization problem:

$$\max_{K_t, N_t} (1 + m_t) A_t F(K_t, N_t) - r_t K_t - w_t N_t.$$  \hspace{1cm} (3)

From the maximization problem of Eq. (3), we have the following first-order conditions:

$$r_t = (1 + m_t) A_t F_1(K_t, N_t).$$  \hspace{1cm} (4)

$$w_t = (1 + m_t) A_t F_2(K_t, N_t).$$  \hspace{1cm} (5)

In this paper, we assume that the firm makes decisions on period-\(t\) quantities of capital and labor in period \(t - 1\). However, following conventional macroeconomic models, we can also assume that the capital amount in period \(t\) is determined in the capital market operating in period \(t - 1\), while the labor supply in period \(t\) is determined in the current period. If instead we let period-\(t\) wage to be determined in the current period, the equilibrium wage would be \(A_t F_2(K_t, N_t)\). Even in this case, the main results of this paper do not change due to perfectly-inelastic labor supply. This would imply that the firm’s profit decreases but the consumer’s labor income increases, and the two effects exactly offset each other and the sum of consumer’s labor, capital and dividend incomes would be \(A_t F(K_t, N_t) + (1 - d) K_t\) in period \(t\). Therefore, there would be no change in consumption-savings decisions.

2.3. The equilibrium

The equilibrium of the economy is characterized by the hyperbolic consumer’s maximization problems in Eqs. (1) and (2), given \((r_t, w_t, \pi_t)^2\); the firm’s maximization problems in Eq. (3), given \((r_t, w_t)^2\); and the labor, capital, and commodity market clearing conditions. The capital market clearing condition is implicit, because we use the same symbol \((k_t)\) for both consumer’s savings and firm’s capital investment. The commodity market clearing condition is satisfied because the aggregate output, \(N_t f_t(k_t)\), is the same as the consumer’s aggregate income, \(N_t (r_t k_t + (1 - d) k_t + w_t)\), by Eq. (4) and (5). Without loss of generality, we have assumed that the labor supply is fixed at \(N_t\) in period \(t\), which implies the following labor market clearing condition:

$$N_t = N_t, \text{ for all } t = 0, 1, 2,$$

and further implies that the real rental rate of capital, real wages, and profits are given by\(^6\)

\(^6\)We allow the firm to have wrong information on its future productivity, which results in negative economic profits. However, this does not mean that its accounting profit is also negative. Economic profits, as considered in most macroeconomic models, include opportunity cost in the form of capital cost in our model. On the other hand, accounting profits, which are mostly reported on corporate balance sheets, do not include this opportunity cost. Therefore, the economic profit is negative but the accounting profit would still be positive if we assume that the firm possesses ownership of capital,
\[ r_t = (1 + m_t)f_t'(k_t), \text{ for all } t = 0, 1, 2, \quad (6) \]
\[ w_t = (1 + m_t)f_t(k_t) - (1 + m_t)k_t f_t'(k_t) \text{ for all } t = 0, 1, 2, \quad (7) \]
and
\[ \pi_t = -m_t f_t(k_t) \text{ for all } t = 0, 1, 2. \quad (8) \]

where
\[ f_t(k_t) = A_t F(K_t, N_t)/N_t \quad \text{and} \quad k_t = K_t/N_t. \]

The optimism parameter \((m_t)\) affects the inverse demand function of capital and labor, as shown in Eqs. (6) and (7), which can be derived from the first-order conditions of the profit maximization in period \(t - 1\). In period \(t\), the production is determined by the real productivity \(A_t\) rather than \((1 + m_t)A_t\), which results in a decrease in profit as described in Eq. (8).

In the economy, there exists an equilibrium, which is shown in the following lemma:

**Lemma 1** There exists an open set \(M \subset \mathbb{R}\) such that \(M \ni 0\) and for any \((m_1, m_2) \in M^2\), there exists an equilibrium of this economy.

Lemma 1 suggests that there exists an equilibrium for no optimism, i.e., \((m_1, m_2) = (0, 0)\) and for moderate levels of pessimism and optimism. The equilibrium satisfies the consumer’s intrapersonal subgame described in subsection 2.1 and the firm’s profit maximization problems in subsection 2.2. However, for values of optimism, \((m_1, m_2)\), that are too large, the consumer’s dividend income can be very low, which results in negative income and thus nonexistence of an equilibrium.

### 3. One-time optimism, capital level, and welfare

A challenge of incorporating optimism in the multi-period model is that one-period’s optimism affects all periods’ commodity and asset prices, as well as the associated decisions of consumers and firms. Therefore, we need to individually assess the impact of each period’s optimism on the equilibrium outcome. In the following Proposition, we first investigate how period-0 optimism about period 1’s productivity affects the equilibrium capital level and welfare:

**Proposition 1** At the equilibrium \((m_1, m_2) = (0, 0)\), a marginal increase in \(m_1\) increases both \(k_1\) and \(k_2\) in equilibrium. This results in a decrease for the period-0 intertemporal utility but increase in the future intertemporal utilities under both ex-
ponential (or $\beta = 1$) and hyperbolic (or $\beta < 1$) discounting, so we have
\begin{align}
\frac{dU(0)}{d m_1}\big|_{(m_1, m_2) = (0, 0)} &< 0, \\
\frac{dU(1)}{d m_1}\big|_{(m_1, m_2) = (0, 0)} &> 0 \text{ and } \frac{dU(2)}{d m_1}\big|_{(m_1, m_2) = (0, 0)} > 0.
\end{align}

Proposition 1 indicates that the capital levels of $(k_1, k_2)$ are boosted by an increase in optimism. Optimism about period-1 ($m_1$) increases the equilibrium real interest rate in period 1. The higher interest rate makes the period-0 consumer save more and thus increases the capital level in period 1 ($k_1$). The increase in the real interest rate has an effect equivalent to an increase in compensated interest rate, because the producer’s optimism decreases the consumer’s dividend income. In other words, the firm’s misestimation of future productivity decreases its profits, which is translated into lower dividend income for consumers. Because the lower dividend income exactly offsets the increased savings income, only the substitution effect plays a role in the consumption-saving decisions.\footnote{It is still controversial in empirical studies whether an increase in the uncompensated real interest rate increases savings (see Gupta 1987). Theoretically, if the intertemporal elasticity of substitution is lower (higher) than one, the uncompensated interest rate would be positively (negatively) correlated with savings. However, it is apparent that the consumer will increase savings as the compensated real interest rate increases, which only contains the substitution effect but not the income effect.}

The higher period-1 capital level ($k_1$) increases the consumer’s savings income in period 1. The consumer will also save more in period 2, as determined by the savings response function, $k_2(k_1)$. Through higher capital levels, period 1 and 2’s intertemporal utilities increase. However, Proposition 1 also indicates that period-1 optimism actually decreases the period-0 intertemporal utility. The current consumer maximizes her utility, which is rational based on the current consumer’s perspective but is present-biased based on the future consumer’s perspective. Therefore, current optimism, which distorts current savings, would make the current consumer worse off while other periods’ consumers better off. This implies that a one-off type of optimism cannot Pareto-improve the economy. Instead, optimism over multiple periods, i.e. sustained optimism, is necessary.

The result in Proposition 1 that current optimism deteriorates current intertemporal utility but improve future intertemporal utilities holds for both time-consistent and -inconsistent models. On the other hand, the impact of future optimism on intertemporal utilities are distinct for two models. The following Proposition shows that future optimism, i.e., the firm’s optimistic view in period 1 on period-2’s productivity, can affect the capital level in a way that benefits the period-0 consumer only in the case where the consumer is present biased. Based on period-0 consumer’s perspective, the period-1 consumer has an undersaving problem under hyperbolic discounting. Therefore, future optimism, which increases period-1 consumer’s equi-
Proposition 2 At the equilibrium \((m_1, m_2) = (0, 0)\), a marginal increase in \(m_2\) increases the period-2 capital level \((k_2)\). It also increases the intertemporal utility in period 0 under hyperbolic discounting (or \(\beta < 1\)). That is, at the equilibrium we have

\[
\frac{dU^{(0)}}{dm_2}|_{(m_1, m_2) = (0, 0)} > 0 \text{ if } \beta < 1.
\]  

(10)

However, the intertemporal utility in period 0 is not increased by optimism \((m)\) under exponential discounting (or \(\beta = 1\)). That is, at the equilibrium we have

\[
\frac{dU^{(0)}}{dm_2}|_{(m_1, m_2) = (0, 0)} = 0 \text{ if } \beta = 1.
\]  

(11)

Proposition 2 shows that an increase in optimism about period 2 increases the equilibrium real interest rate in period 2 and thus also increases the period-2 savings level \((k_2)\) for any given \(k_1\). This increase in the future savings \((k_2)\) can increase the period-0 intertemporal utility if and only if the consumer exhibits hyperbolic decision-making (or \(\beta < 1\)). Under hyperbolic discounting, the current self (in period 0)’s welfare is negatively affected by the future self (in period 1)’s undersavings problem. Therefore, future optimism \((m_2)\) increases future savings \((k_2)\), which positively impacts period-0 intertemporal utility.

Propositions 1 and 2 both indicate that optimism helps improve the other periods’ intertemporal utilities under hyperbolic discounting but not that of the current period. Therefore, to generate Pareto-improvement, the firm has to maintain sustained optimism, which implies that the firm has optimistic views about the future in all periods. From Propositions 1 and 2, we can show the existence of Pareto-improving optimism.

4. The impact of optimism and pessimism on welfare

It is well-known that an economy with hyperbolic consumers experiences an undersaving problem.\footnote{See Strotz (1956); Phelps and Pollak (1968); Laibson (1996, 1997); Kang (2019).} This means that an increase in all periods’ savings from the equilibrium level would improve all periods’ intertemporal utilities. This section shows that firm optimism can increase the equilibrium capital level and has the potential to generate Pareto-improvement in the economy. Where there is no optimism, i.e. \((m_1, m_2) = (0, 0)\), there exists an equilibrium satisfying the first and second order
conditions. This implies that for an infinitesimal increase in optimism, an equilibrium still exists. This section investigates how intertemporal utilities change given infinitesimal changes in optimism.

A difficulty with resolving the undersaving problem from present-bias is that not every marginal increase in investment guarantees Pareto-improvement of the equilibrium allocations. The level of capital should increase above the savings response function $k_2(k_1)$ to increase current intertemporal utility, i.e., the incremental capital amount should be inside the Pareto-superior area. The current consumer maximizes the current intertemporal utility given the savings response function and thus, the indifference curve of the current intertemporal utility would be tangent to the savings response function. Therefore, even if the capital levels in both periods increase, current intertemporal utility could nevertheless decrease if the increase is along or outside of the savings response function.

This suggests that although producers’ optimism affects the current consumers’ decisions, it cannot improve current-period utility, as shown in Proposition 1. To improve current intertemporal utility, some degree of future optimism is necessary to affect the saving response function. Analogous logic applies to the future consumers. Future-period optimism does not improve future consumer’s welfare. Only optimism in the past, which increases the earlier-period savings, would be beneficial to the future consumer as shown in Proposition 2. In sum, a combination of current and future optimism is necessary to Pareto-improve the economy, as will be shown in the following section.

**Proposition 3** There exists an optimism parameter set $(m_1, m_2) > 0$ that improves all intertemporal utilities under hyperbolic discounting (or $\beta < 1$).

Proposition 3 indicates that having optimistic views in both periods can Pareto-improve the equilibrium allocations. With only optimism about period 1, $U^{(1)}$ and $U^{(2)}$ improve but not $U^{(0)}$. However, together with optimism about both periods, all intertemporal utilities can improve.

With the reverse logic in Proposition 3, we can show that sustained pessimism decreases all intertemporal utilities, as shown in the following Proposition:

**Proposition 4** There exists a pessimism parameter set $(m_1, m_2) < 0$ that decreases all intertemporal utilities under hyperbolic discounting (or $\beta < 1$).

## 5. Steady-state analysis

In this section, we introduce an infinite period model in which a unique steady-state equilibrium exists. The three-period model is useful in understanding the theoretical mechanism of how optimism and present bias jointly affect the consumer and firm’s investment decisions. Specifically, we have shown in Section 3 that current and
future optimism affect the intertemporal utilities in different ways: current optimism increases future intertemporal utilities but not current utilities, while future optimism affects welfare in the opposite way. However, the three-period model in Section 3 is more limited for showing the quantitative macroeconomic impact of optimism. To overcome this limitation, we introduce a steady-state transition analysis with uniform optimism, which means that both the current and future consumers have the same degree of optimism. With reasonable macroeconomic parameter values, we quantify the individual and joint impact of optimism and present bias on the steady-state capital-to-output ratio. We also quantify the welfare gain of the economy under optimism. For the welfare analysis, we assume that the economy in period 0 is in the steady state without optimism.

In this section, we assume that the firm permanently has optimistic views on productivity. This assumption might be limited in the sense that the firm may not indefinitely misjudge its optimal decision in the long run. However, the main focus of this section is to understand the quantitative impact of optimism in the hyperbolic economy rather than the question of whether the government can implement optimism-enhancing policies in the long run. The quantitative result in this section shows that producer optimism is an effective way to improve welfare (specifically, 7% permanent optimism can improve the welfare equivalent to 4~43% increase in consumption), which suggests that the optimism-enhancing policy can be effective even if adopted with a short-horizon.

5.1. The multi-period model

This subsection introduces a multi-period model in which we incorporate firm optimism into a steady state model, in the spirit of Laibson (1997). The representative consumer’s budget constraint in year $t$ is

$$c_t + k_{t+1} = r_t k_t + (1 - d)k_t + w_t + \pi_t,$$

(12)

where $d$ is the capital depreciation rate, $r_t$ is the real rental rate of capital, $w_t$ is the real wage, and $\pi_t$ is the dividend income in period $t$. We also assume that the representative consumer inelastically supplies one unit of labor. The consumer lives $T$ periods, but $T$ can be extended to infinite periods in steady-state analysis. Period-$t$’s intertemporal utility under quasi-hyperbolic discounted time preferences is

$$U(t)(c_0, c_1, \ldots, c_T) = u(c_t) + \beta \sum_{i=1}^{T-t} \delta^i u(c_{t+i}),$$

(13)

where we assume $u(c)$ is the CES instantaneous utility function:

$$u(c) = \begin{cases} \frac{c^{1-\rho} - 1}{1-\rho} & \text{if } \rho > 1, \\ \ln c & \text{if } \rho = 1. \end{cases}$$
We consider a standard Cobb-Douglas production function, where $K_t$, $N_t$, and $A_t$ represent aggregate capital, aggregate labor, and exogenous productivity, respectively:

$$Y_t = A_t K_t^\alpha N_t^{1-\alpha}.$$ 

The optimistic firm believes, in period $t-1$, that the period-$t$ production function is

$$Y^{(m)}_t = (1 + m) A_t K_t^\alpha N_t^{1-\alpha},$$

where $m$ represents uniform optimism, such that $m = m_1, \ldots, = m_T$. The demand for capital is determined by the firm’s subjective production function instead of the actual production function. Thus, we have the following equilibrium gross real interest rate in period $t$:

$$r_t = \frac{Y_t}{K_t} (1 + m). \quad (14)$$

Eq. (14) indicates that the producer’s optimistic view increases the real interest rate of the economy.

By the dynamic budget constraint and the capital market clearing conditions, in equilibrium we have

$$r_t k_t + w_t + \pi_t = f_t(k_t), \quad (15)$$

$$r_t = (1 + m) f'_t(k_t), \quad (16)$$

$$w_t = (1 + m) f_t(k_t) - (1 + m) k_t f'_t(k_t), \quad (17)$$

and

$$\pi_t = -m f_t(k_t).$$

where $f_t(k_t)$ is the per-capita production function in year $t$.

### 5.2. Steady-state capital-to-output ratio

In the rational expectation model, the consumer can perfectly forecast not only the future interest rates but also future capital, labor and dividend incomes. In the economy under quasi-hyperbolic time preferences, the Euler equation is characterized by

$$u'(c_t) = (r_{t+1} + 1 - d) \delta u'(c_{t+1}) [1 - (1 - \beta) \Lambda_{t+1}], \quad (18)$$

where $\Lambda_{t+1} \in (0, 1)$ represents the marginal consumption with respect to wealth in period $t+1$. For the derivation of the Euler equation, see Appendix F.

In the hyperbolic economy, there can be multiple equilibria (see Laibson 1994). To overcome this multiplicity, Laibson (1994, 1997) suggested the sufficient condition for the existence of unique equilibrium in the finite T-period model and attains uniqueness is to look at the limit of the equilibrium of finite T-period economies as $T$ goes to infinity. Given that our Euler equation from consumer’s maximization problem in Eq. (18), even under optimism ($m > 0$) would have the same format as Laibson (1997),
the sufficient condition in Laibson (1997) can be directly applied to our model. That is
\[ u'(w_t) > \beta \delta^\tau \left( \prod_{i=1}^{\tau} (r_{t+i} + 1) \right) u'(w_{t+\tau}) \quad \text{for all } t \text{ and } \tau. \]  

(19)

The condition in Eq. (19) implies that the future labor income \( w_{t+\tau} \) is large enough so that there would be no corner solutions in future savings decisions. Laibson (1997) shows that the condition in Eq. (19) is satisfied for a wide range of parameters \( \beta \) and \( \rho \). We assume that the condition in Eq. (19) is satisfied, so the unique equilibrium exists. In the numerical analysis, we check the condition in Eq. (19) and confirm that the condition is satisfied.

If \( \beta = 1 \), Eq. (18) becomes a typical Euler equation with exponential discounted time preference. As \( \beta \) decreases (i.e., the consumer suffers more from time-inconsistency), \( [1 - (1 - \beta) \Lambda_{t+1}] \) decreases and therefore, the equilibrium interest rate decreases. In the economy with hyperbolic discounting and optimism, consumption in each period can be characterized by the following rule:

\[ c_t = \lambda_{t-1} W_t, \]

where \( W_t \) is the sum of financial asset, the discounted value of future labor, and dividend incomes. Then, we have \( \Lambda_{t+1} = \lambda_{t-1} \). From Eq. (18) and \( c_t = \lambda_{t-1} W_t \), we have the sequence \( \{\lambda_t\}_{t=0}^T \) given by the recursion,

\[ \lambda_{i+1} = \frac{\lambda_i}{\delta (r_{i+1} + 1 - d)^{1-\rho} (\lambda_i (\beta - 1) + 1)^{1/\rho}} + \lambda_i \quad \text{and} \lambda_0 = 1. \]  

(20)

When \( T \) is arbitrarily large (\( T \to \infty \)), the real rental rate of capital converges to the steady-state interest rate \( r^* \) and \( c_t \) converges to \( \lambda^* W_t \), where we have \( \lambda^* = \lambda_i = \lambda_{i+1} \). \( \lambda^* \) can be derived from Eq. (20):

\[ \lambda^* = 1 - \delta \frac{1}{\delta} (r^* + 1 - d)^{1-\rho} [1 - \lambda^* (1 - \beta)]^{1/\rho}, \]  

(21)

and the Euler equation in Eq. (18) converges to

\[ u'(c_t) = (r^* + 1 - d) \delta u'(c_{t+1}) (\lambda^* (\beta - 1) + 1). \]  

(22)

\( A_t \) is assumed to grow exogenously at the rate \( g_A \). Therefore, in the steady state, capital and output must grow at the rate \( g_A/(1-\alpha) \equiv g \). With proportional consumption, the steady state condition is

\[ (r^* + 1 - d) (1 - \lambda^*) = \exp(g). \]  

(23)

From Eq. (14), (22) and (23), we have the following steady-state equilibrium with the hyperbolic-discounting parameter \( \beta \) and the optimism parameter \( m \):

**Proposition 5** There exists a unique steady state satisfying

\[ \frac{K^*}{Y^*} = \frac{(1 + m)\alpha \beta \delta}{\exp(\rho g) - \delta \exp(g) (1 - \beta) - (1 - d) \beta \delta}, \]  

(24)
where $Y^*$ and $K^*$ are the steady-state output and capital levels, respectively.

Proposition 5 indicates that the steady-state capital-to-output ratio increases in optimism ($m$), while it decreases as the consumer becomes more present-biased (i.e., $\beta$ decreases). With $\alpha = 0.36, d = 0.08, g = 0.02, \beta = 1, K/Y = 3$ and when the firm has no cognitive bias (i.e., $m = 0$), we have the following equation from Eq. (24):

$$\exp(\rho(0.02)) = \delta(1.04).$$

Substituting Eq. (25) into Eq. (24), we have

$$\frac{K^*}{Y^*} = \frac{(1 + m)0.36\beta}{1.04 - \exp(0.02)(1 - \beta) - 0.92\beta}$$

which is independent of $\rho$ and $\delta$. From Eq. (26), we know how the value of $m$ affects the steady-state capital-output ratio. For example, for a value of $m$ being 3% (7%), the capital-output ratio is higher by 2.8% (7.1%) in the economy with hyperbolic discounting (i.e., $\beta = 0.7$) relative to the no present bias case, as shown in Table 1. The value of $\beta$ for individual consumers is estimated to be around 0.7, based on experimental and field evidence (see Angeletos et al. 2001 and Laibson, Repetto, and Tobacman 2007).\footnote{In the seminal macroeconomics application of hyperbolic discounting by Laibson (1997), he chooses $\beta = 0.6$ for quantitative analysis. However, subsequent papers with the hyperbolic economy, including others by Laibson, have chosen the value of $\beta$ as 0.7.}

<table>
<thead>
<tr>
<th>$m$</th>
<th>0%</th>
<th>3%</th>
<th>5%</th>
<th>7%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.00</td>
<td>3.09</td>
<td>3.15</td>
<td>3.21</td>
</tr>
<tr>
<td>0.9</td>
<td>2.95</td>
<td>3.03</td>
<td>3.09</td>
<td>3.15</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.8</td>
<td>2.88</td>
<td>2.97</td>
<td>3.03</td>
</tr>
<tr>
<td>0.7</td>
<td>2.80</td>
<td>2.88</td>
<td>2.94</td>
<td>\textbf{3.00}</td>
</tr>
<tr>
<td>0.6</td>
<td>2.70</td>
<td>2.78</td>
<td>2.84</td>
<td>2.89</td>
</tr>
</tbody>
</table>

Table 1. Steady-state capital-to-output ratio

Table 1 indicates that optimism is quite effective in recovering the capital loss due to present bias. In the typical hyperbolic discounted macroeconomic setting, only 7% of optimism is necessary to transform the hyperbolic economy to the economy without present bias. Specifically, the economic variables; including consumption, output, and savings; in the economy with neither present bias nor cognitive bias (i.e., $(\beta, m) = (1,0)$), are the same as those with 30% of present bias and 7% of optimism (i.e., $(\beta, m) = (0.7,7\%)$).

This subsection shows that optimism can be effective in increasing the steady-state capital level. However, a remaining question is whether optimism also improves
welfare. In the following subsection, we investigate how the consumer’s intertemporal utilities change during the transition from the steady-state without optimism to that under optimism.

5.3. Steady-state transitions and welfare

Whether the “initial” consumer (period-1 consumer) can be better off under optimism bears important policy implications. For the government to induce optimism in period 1, it might need to generate consensus from the current consumer, which is the period-1 consumer. Even if this policy helps future consumers, it may be politically difficult to adopt the policy if it hurts the current consumers. However, this paper shows that with reasonable macroeconomic parameters, sustained optimism clearly improves both the initial and future consumer’s welfare.

For the welfare analysis, we need to simulate the equilibrium transition from the steady-state without optimism \((m = 0\%)\) to the other steady-state with optimism (e.g., \(m = 7\%\)). The numerical methodology for finding the convergence path when optimism enters the economy in year 1 is as follows.\(^{10}\) First, we need to guess the equilibrium consumption \(c_1\). Based on \(c_1\), we can derive \(k_2\) from the market clearing condition of Eq. (12). From the capital market equilibrium condition, that is, \(r_t = \alpha Y_t/K_t(1 + m)\), we can derive \(r_t\) from \(k_t\). \(c_2\) can be derived from the Euler equation of Eq. (18). In this way, we can derive a sequence of \(c_t\)\(t=1\). If the sequence \(c_t\)\(t=1\) is not converging to the steady state, which is characterized by Eq. (24), we repeat the search process with another guess of \(c_1\).\(^{11}\)

We choose the value of 0.7 for the hyperbolic discounting factor \(\beta\). We set \(\rho = 1\) and \(\delta = 0.981\) in this subsection. The parameter choices of \((\rho, \delta)\) do not affect the steady state capital-to-output level but influence the speed of convergence. The other parameter choices are the same as those in subsection 5.2. Figure 1 shows that the equilibrium capital-to-output ratio changes over time when 7% of optimism enters the economy in year 1. In year 0, the economy is in the steady-state under no optimism \((m = 0\%)\). However, starting in period 1, the consumer increases savings and the capital-to-output ratio is converging to 3.

We measure the intertemporal utility loss or gain in terms of consumption goods. When the consumer in period \(t\) is indifferent between a consumption plan without optimism plus a “virtual” one-time additional consumption subsidy \((c_t' h_t)\), as compared to a consumption plan with optimism, the percentage of the additional consumption good \((h_t)\) is the measure of welfare gain. Specifically, the welfare measure in period

\(^{10}\)A similar type of numerical methodology has been introduced in Laibson (1997, page 466).

\(^{11}\)The numerical analysis in this subsection was performed with MATLAB 9. All MATLAB codes can be downloaded from mwkang.site11.com/code/macro_optimism.
Steady state without optimism
Steady state with optimism (m=0.07)
Capital-to-
output
ratio
Year
0 10 20 30 40 50 60
2.8
2.85
2.9
2.95
3
Figure 1: Equilibrium capital-output ratio changes over years when 7% of sustainable optimism starts in year 1 under the economy with $\alpha = 0.36$, $d = 0.08$, $g = 0.02$, $\rho = 1$, and $\delta = 0.981$.

\begin{equation}
t\left(h_t\right) \text{ is one that satisfies the following equation:}
\end{equation}

\begin{equation}
\begin{aligned}
&u(c_t^* + c_t^* h_t) + \beta \sum_{\tau=1}^{\infty} \delta^\tau u(c_{t+\tau}^*) = u(c_t^+) + \beta \sum_{\tau=1}^{\infty} \delta^\tau u(c_{t+\tau}^+), \\
&\text{where } (c_t^*, c_{t+1}^*, \ldots) \text{ is the equilibrium consumption plan without optimism and } (c_t^+, c_{t+1}^+, \ldots) \text{ is the one with optimism. Optimism would be Pareto-improving if } h_t > 0 \text{ for all } t \geq 1.
\end{aligned}
\end{equation}

The numerical simulation shown in Figure 2 shows that a 7% optimism on expected productivity is a clear Pareto-improvement relative to the equilibrium allocations.\footnote{Not shown in this paper, we conduct other numerical analyses under varying optimism parameters, such as $m = 1\%, 2\%, 3\%, \ldots, 7\%$. In all of these cases, the equilibrium allocations are Pareto-improving, i.e., $h_t > 0$ for all $t = 1, 2, 3, \ldots$.}

The welfare gain ($h_t$) is strictly increasing over years, because the future consumer will benefit from the increased capital accumulation. Year-1 consumer’s welfare gain is smallest among all years’ consumers but still as large as 4% (i.e., $h_1 = 4\%$). The welfare gain of “future” consumer, as measured by additional consumption goods, in the new steady-state amounts to a sizable 43% (i.e., $h_{\infty} = 43\%$). The quantitative welfare gains as analyzed in this framework lend support to the idea that optimism can be beneficial to the economy.

6. Conclusion

We present a framework that assesses the effect of optimism under a setting of present bias. Our framework sheds light on how optimism can play a positive role in
the macroeconomic setting despite its value-reducing nature in the corporate setting. In a three-period framework under present-biased consumers, we show that optimism can be effective in mitigating the undersaving problem resulting from present bias. Furthermore, the type of optimism that generates this Pareto-improvement must be sustained over multiple periods, rather than temporary. This results from the notion that optimism in any given period boosts capital in the future period but decreases welfare in the current period. In our multi-period steady state analysis, we show that optimism in the context of our framework is quantitatively relevant. Both the qualitative and quantitative results suggest that fostering optimistic sentiments to the economy can be effective under some circumstances.

In this paper, we have focused on producer optimism but not consumer optimism. In the academic and policy literature, consumer optimism/pessimism is also an important element in the macroeconomy. For example, Abel (2002) and Giordani, Söderlind (2006) analyzed the impact of consumer’s pessimism on stock and bond prices in the macroeconomy. Abel (2002) defined consumer’s optimism (pessimism) as the consumer’s expectation about future real interest rate being higher (lower) than the actual interest rate. Under this setting, the intertemporal elasticity of substitution has an important role in determining whether the consumer can save more with optimism. Only in the case where the intertemporal elasticity of substitution is higher than 1 could optimism increase savings. On the other hand, we show that the producer optimism always increases equilibrium investment (savings) for any value of elasticity of substitution. Government actions can influence both producer optimism and consumer optimism, but at least under our main example with log-linear preferences, consumer optimism does not have any impact on saving decisions. For non-log-linear preferences, consumers’ optimism/pessimism could have real impact on the economy, and further studies should be conducted in this area.
Appendices

A. Proof of Lemma 1

The capital market clearing condition is satisfied because we use the same symbol \((k_t)\) for both consumer’s savings and firm’s capital investment. The commodity market clearing condition is automatically satisfied because the aggregate output, \(N_t f_t(k_t)\), is the same as the consumer’s aggregate income, \(N_t (r_t k_t + (1 - d) k_t + w_t)\) by the following first-order conditions from the firm’s profit maximization problem:

\[
 r_t = (1 + m_t) f_t'(k_t), \text{ for all } t = 0, 1, 2; \tag{28}
\]

\[
 w_t = (1 + m_t) f_t(k_t) - (1 + m_t) k_t f_t'(k_t) \text{ for all } t = 0, 1, 2, \tag{29}
\]

and

\[
 \pi_t = -m_t f_t(k_t) \text{ for all } t = 0, 1, 2. \tag{30}
\]

The remaining part of the proof is to show that there exists optimal consumption-savings level for any given \(r_t > 0, w_t > 0\), and \(\pi_t \in (- (r_t k_t + (1 - d) k_t + w_t), \infty)\). The lower boundary limit \((- (r_t k_t + (1 - d) k_t + w_t)\) for \(\pi_t\) ensures the consumer to have a positive income in period \(t\). Since \(\pi_t = 0\) if \(m_t = 0\) from Eq. (30), we know that there is an open set \(M \subset \mathbb{R}\) and \(M \in \{0\}\) for any \((m_1, m_2) \subset M^2\), the consumer’s equilibrium incomes for all three periods are strictly positive.

In period 1, the representative consumer chooses \(k_2\) to maximize its period-2 utility function given any \(k_1 > 0\):

\[
 \max_{k_2 \mid k_1} u(c_1) + \beta \delta u(c_2) \tag{31}
\]

subject to

\[
 c_1 = r_1 k_1 + (1 - d) k_1 + w_1 + \pi_1 - k_2,
\]

\[
 c_2 = r_2 k_2 + (1 - d) k_2 + w_2 + \pi_2.
\]

The first order condition of the maximization problem of Eq. (31) is

\[
 -u'(c_1) + \beta \delta u'(c_2) (r_2 + 1 - d) = 0. \tag{32}
\]

The second order condition from the maximization problem of Eq. (31) is

\[
 u''(c_1) + \beta \delta u''(c_2) (r_2 + 1 - d)^2 < 0. \tag{33}
\]

By the first and second order conditions of Eqs. (32) and (33), we know that for any value of \(k_1 > 0\), there exists a unique \(k_2 > 0\) that solves Eq. (32). We define \(k_2(k_1)\),
which solves the first order condition in (32), such that
\[-u'(r_1 k_1 + (1 - d)k_1 + w_1 + \pi_1 - \overline{k}_2(k_1)) + \beta \delta u'(r_2 \overline{k}_2(k_1) + (1 - d)\overline{k}_2(k_1) + w_2 + \pi_2) (r_2 + 1 - d) = 0.\]  
(34)

Implicitly differentiating Eq. (34) with respect to \(k_1\), we have
\[-u''(c_1) \left( r_2 + (1 - d) - \overline{k}_2(k_1) \right) + \beta \delta u''(c_2) \overline{k}_2(k_1)^2 \overline{k}_2(k_1) + \beta \delta u'(c_2) (1 + m_2) f''_2(k_2) \overline{k}_2(k_1) = 0,\]  
(35)

From Eq. (36), we have
\[
\overline{k}'_2(k_1) = \frac{u''(c_1) (r_2 + 1 - d)}{u''(c_1) + \beta \delta u''(c_2) (r_2 + 1 - d)^2 + \beta \delta u'(c_2) (1 + m_2) f''_2(k_2)} > 0.\]  
(36)

where \(\overline{k}_2(k_1)\) is strictly increasing and continuously differentiable.

Plugging \(\overline{k}_2(k_1)\) into \(U^{(0)}\), we obtain
\[
U^{(0)} = u(r_0 k_0 + (1 - d)k_0 + w_0 + \pi_0 - k_1) + \beta \delta u(r_1 k_1 + (1 - d)k_1 + w_1 + \pi_1 - \overline{k}_2(k_1)) + \beta \delta^2 u(r_2 \overline{k}_2(k_1) + (1 - d)\overline{k}_2(k_1) + w_2 + \pi_2),\]  
(37)

By the market clearing condition, we have \(r_0 k_0 + (1 - d)k_0 + w_0 + \pi_0 = f_0(k_0) + (1 - d)k_0\). By the limiting condition of utility, such that \(\lim_{c \to -\infty} u(c) = -\infty\), we know that the equilibrium capital level \(k_1\) is not bounded, i.e., \(k^*_1 \in (0, f_0(k_0) + (1 - d)k_0)\). This implies that there exists an interior solution \(k^*_1\), which satisfies the following first and second order conditions by the mean value theorem. The sophisticated consumer’s self in period 1 maximizes the utility of Eq. (38) and the first order condition is
\[-u'(c_0) + \beta \delta u'(c_1) \left( r_1 + (1 - d)k_1 - \overline{k}_2(k_1) \right) + \beta \delta^2 u'(c_2) (r_2 + 1 - d) \overline{k}_2(k_1) = 0.\]  
(38)

The first order condition is the necessary condition for the maximization of Eq. (38). The necessary second order condition at the equilibrium capital choice is
\[
u''(c_0) + \beta \delta u''(c_1) \left( r_2 + 1 - d - \overline{k}_2(k_1) \right)^2 - \beta \delta u'(c_1) \overline{k}_2(k_1) + \beta \delta^2 u''(c_2) \left( r_2 + 1 - d \right) \overline{k}_2(k_1)^2 + \beta \delta^2 u'(c_2) (r_2 + 1 - d) \overline{k}_2(k_1) \leq 0.\]  
(39)
Proof of Proposition 1

In the maximization problem in period 0 with a choice function $k_2(k_1)$, we have the following first order condition:

$$-u'(c_0) + \beta \delta u'(c_1) \left( r_1 + (1 - d)k_1 - \bar{k}_2(k_1) \right) + \beta \delta^2 u'(c_2) (r_2 + 1 - d) \bar{k}_2(k_1) = 0 \quad (41)$$

where

$$c_0 = r_0 k_0 + (1 - d)k_0 + w_0 + \pi_0 - k_1,$$
$$c_1 = r_1 k_1 + (1 - d)k_1 + w_1 + \pi_1 - k_2,$$
$$c_2 = r_2 k_2 + (1 - d)k_2 + w_2 + \pi_2.$$  

Taking the total derivative of Eq. (41) with respect to $k_1$, we have

$$SD_0 = u''(c_0) + \beta \delta u''(c_1) \left( r_2 + 1 - d - \bar{k}'_2(k_1) \right)^2$$
$$+ \beta \delta u'(c_1) \left( (1 + m_1) f''_1(k_2) - \bar{k}''_2(k_1) \right) + \beta \delta^2 u''(c_2) \left( (r_2 + 1 - d) \bar{k}_2'(k_1) \right)^2$$
$$+ \beta \delta^2 u'(c_2) (1 + m_2) f''_2(k_2) \left( \bar{k}'_2(k_1) \right)^2 + \beta \delta^2 u'(c_2) (r_2 + 1 - d) \bar{k}_2'(k_1) \quad (42)$$

From the inequality of Eq. (40), $f''_1(k_2) < 0$ and $f''_2(k_2) < 0$, we know that the second-order condition $(SD_0)$ in Eq. (42) is strictly negative. Where $(m_1, m_2) = (0, 0)$, the second order condition in Eq. (42) is equivalent to Eq. (40). Implicitly differentiating the FOC of Eq. (41) in terms of $m_1$, we have

$$SD_0 \times dk_1$$
$$+ \beta \delta u''(c_1) \left( r_1 k_1 + (1 - d) - \bar{k}'_2(k_1) \right) \left( \frac{dr_1}{dm_1} k_1 + \frac{d\pi_1}{dm_1} \right)$$
$$+ \beta \delta u'(c_1) \frac{dr_1}{dm_1} = 0. \quad (43)$$

Because we have $\frac{dr_1}{dm_1} k_1 + \frac{dr_1}{dm_1} = 0$ and $\frac{dr_1}{dm_1} > 0$, where $(m_1, m_2) = (0, 0)$, from Eq. (43) we have

$$\frac{dk_1}{dm_1} = -\frac{\beta \delta u'(c_1) \frac{dr_1}{dm_1}}{SD_0} > 0. \quad (44)$$

Because period-1 optimism $(m_1)$ does not change the savings response function $\bar{k}_2(k_1)$, we have the following equality:

$$\frac{dk_2}{dm_1} / \frac{dk_1}{dm_1} = \bar{k}'_2(k_1). \quad (45)$$
The period-0 intertemporal utility is

\[
U(0) = r_0k_0 + (1 - d)k_0 + w_0 + \pi_0 - k_1,
\]

\[
r_1k_1 + (1 - d)k_1 + w_1 + \pi_1 - k_2(k_1),
\]

\[
r_2k_2(k_1) + (1 - d)k_2(k_1) + w_2 + \pi_2.
\]

(46)

Applying the envelope theorem to Eq. (46), we have

\[
\frac{dU(0)}{dm_1} = \frac{dU(1)}{dk_1} \frac{dk_1}{dm_1} = \frac{\partial U(1)}{\partial c_1} \frac{dk_1}{dm_1} < 0.
\]

(47)

The period-1 intertemporal utility is

\[
U(1) = r_1k_1 + (1 - d)k_1 + w_1 + \pi_1 - k_2, r_2k_2 + (1 - d)k_2 + w_2 + \pi_2.
\]

(48)

Applying the envelope theorem to Eq. (48), we have

\[
\frac{dU(1)}{dm_1} = \frac{dU(1)}{dk_1} \frac{dk_1}{dm_1} = \frac{\partial U(1)}{\partial c_1} (r_1 + 1 - d) \frac{dk_1}{dm_1} > 0.
\]

(49)

The period-2 intertemporal utility is

\[
U(2) = u(r_2k_2 + (1 - d)k_2 + w_2 + \pi_2).
\]

(50)

Applying the envelope theorem to Eq. (50), we have

\[
\frac{dU(2)}{dm_2} = \frac{dU(2)}{dk_2} \frac{dk_2}{dm_1} = \frac{\partial U(2)}{\partial c_2} (r_2 + 1 - d) \frac{dk_1}{dm_1} > 0.
\]

(51)

C. Proof of Proposition 2

In the maximization problem in period 1, we have the following first-order condition:

\[-u'(c_1) + \beta \delta u'(c_2) (r_2 + 1 - d) = 0,\]

(52)

where

\[c_1 = r_1k_1 + (1 - d)k_1 + w_1 + \pi_1 - k_1\]

and

\[c_2 = r_2k_2 + (1 - d)k_2 + w_2 + \pi_2.\]

The second-order condition can be derived from Eq. (52):

\[SD_1 = u''(c_1) + \beta \delta u'(c_2) (r_2 + 1 - d)^2 + \beta \delta u'(c_2) \frac{dr_2}{dk_2} < 0,\]

(53)
where
\[ \frac{dr_2}{dk_2} = (1 + m_2)f''_2(k_2). \]
Implicitly differentiating Eq. (52) in terms of \( m_2 \), we have
\[ SD_1dk_2 + \beta \delta u''(c_2) \frac{dc_2}{dm_2} (r_2 + 1 - d) dm_2 + \beta \delta u'(c_2) \left( \frac{dc_2}{dm_2} \right) dm_2 = 0. \] (54)
We have \( \frac{dc_2}{dm_2} = 0 \), since \( \frac{dr_1}{dm_1}k_1 + \frac{dr_1}{dm_1} = 0 \). Thus, from Eq. (54) we have
\[ \frac{d\bar{k}_2(k_1)}{dm_2} = -\frac{\beta \delta u'(c_2) \left( \frac{dc_2}{dm_2} \right)}{SD_1} > 0. \] (55)
Because we have \( \frac{dc_2}{dm_2} = f'_2(k_2) > 0 \) and \( SD_1 < 0 \), we have \( \frac{d\bar{k}_2(k_1)}{dm_2} > 0 \). Implicitly differentiating \( \bar{k}_2(k_1) = k_2 \) in terms of \( m_2 \), we have
\[ \frac{d\bar{k}_2(k_1)}{dm_2} + \bar{k}_2'(k_1) \frac{dk_1}{dm_2} + \frac{dk_2}{dm_2} = 0. \] (56)
From Eqs. (55) and (56), we have
\[ \bar{k}_2'(k_1) \frac{dk_1}{dm_2} < \frac{dk_2}{dm_2}. \] (57)
The period-0 intertemporal utility is
\[ U^{(0)} \left( \begin{array}{c} r_0k_0 + (1 - d)k_0 + w_0 + \pi_0 - k_1, \\ r_1k_1 + (1 - d)k_1 + w_1 + \pi_1 - \bar{k}_2(k_1), \\ r_2\bar{k}_2(k_1) + (1 - d)\bar{k}_2(k_1) + w_2 + \pi_2 \end{array} \right). \] (58)
Applying the envelope theorem to Eq. (58), we have
\[ \frac{dU^{(0)}}{dk_2} \frac{d\bar{k}_2(k_1)}{dm_2} = \left\{ -\beta \delta u'(c_1) + \beta \delta^2 u'(c_2) (r_2 + 1 - d) \right\} \frac{d\bar{k}_2(k_1)}{dm_2}. \] (59)
From Eq. (32) and equality (59), we have
\[ \frac{dU^{(0)}}{dm_2} = (1 - \beta) \delta u'(c_1)\frac{d\bar{k}_2(k_1)}{dm_2} > 0 \text{ if } \beta < 1, \]
\[ = 0 \text{ if } \beta = 1. \] (60)
D. Proof of Proposition 3

To explain the underinvestment problem in terms of capital level, we define the three intertemporal utilities as functions of \((k_1, k_2)\), such that

\[
\bar{U}^{(0)}(k_1, k_2) = U^{(0)}(f_0(k_0) + (1-d)k_0 - k_1, f_1(k_1) + (1-d)k_1 - k_2, f_2(k_2) + (1-d)k_2), \tag{61}
\]

\[
\bar{U}^{(1)}(k_1, k_2) = U^{(1)}(f_1(k_1) + (1-d)k_1 - k_2, f_2(k_2) + (1-d)k_2), \tag{62}
\]

and

\[
\bar{U}^{(2)}(k_1, k_2) = U^{(2)}(f_2(k_2) + (1-d)k_2). \tag{63}
\]

From the function in Eq. (63), it is clear that \(U^{(2)}\) is strictly increasing in \(k_2\) but is invariant to \(k_1\):

\[
\frac{\partial U^{(2)}}{\partial k_1} = 0 \quad \text{and} \quad \frac{\partial U^{(2)}}{\partial k_2} > 0.
\]

We have the following lemma showing whether \(U^{(0)}\) and \(U^{(1)}\) increase or decrease by the change in capital level at the equilibrium.

**Lemma 2** At the equilibrium investment plan, \((k_1^*, k_2^*)\), if \(\beta \in (0, 1)\) and \((m_1, m_2) = 0\), we have

\[
\frac{\partial U^{(0)}}{\partial k_1} < 0, \quad \frac{\partial U^{(0)}}{\partial k_2} > 0, \quad \frac{\partial U^{(1)}}{\partial k_1} > 0 \quad \text{and} \quad \frac{\partial U^{(1)}}{\partial k_2} = 0.
\]

**Proof.** Taking the partial derivative of \(U^{(0)}\) with respect to \(k_1\) at equilibrium \((k_1^*, k_2^*)\), we have

\[
\frac{\partial U^{(0)}}{\partial k_1} \bigg|_{(k_1, k_2) = (k_1^*, k_2^*)} = -u'(c_0) + \beta \delta u'(c_1)(f_1'(k_1) + 1 - d). \tag{64}
\]

From Eq. (39), we have

\[
\begin{align*}
- u'(c_0) + \beta \delta u'(c_1)(r_1 + 1 - d) \\
= (\beta \delta u'(c_1) - \beta \delta^2 u'(c_2)(f_2'(k_2) + 1 - d) ) \bar{F}'_2(k_1) \\
= \beta \delta (u'(c_1) - \delta u'(c_2)(f_2'(k_2) + 1 - d) ) \bar{F}'_2(k_1). \tag{65}
\end{align*}
\]

From Eq. (32), we have

\[
u'(c_1) = \beta \delta u'(c_2)(f_2'(k_2) + 1 - d). \tag{66}
\]

From Eqs. (65) and (66), we have

\[
\frac{\partial U^{(0)}}{\partial k_1} \bigg|_{(k_1, k_2) = (k_1^*, k_2^*)} < 0.
\]
Taking the partial derivative of $U^{(0)}$ with respect to $k_2$ at the equilibrium $(k_1^*, k_2^*)$, we have
\[
\frac{\partial U^{(0)}}{\partial k_2} \bigg|_{(k_1,k_2)=(k_1^*,k_2^*)} = -\beta \delta u'(c_1) + \beta \delta^2 u'(c_2)(f'_2(k_2) + 1 - d) \\
= -\beta \delta (-u'(c_1) + \delta u'(c_2)(f'_2(k_2) + 1 - d)). \quad (67)
\]

At the equilibrium, the first order condition of Eq. (32) is
\[
-u'(c_1) + \beta \delta u'(c_2)(f'_2(k_2) + 1 - d) = 0. \quad (68)
\]

From Eqs. (67) and (68), we have
\[
\frac{\partial U^{(0)}}{\partial k_2} \bigg|_{(k_1,k_2)=(k_1^*,k_2^*)} > 0.
\]

Taking partial derivative $U^{(1)}$ with respect to $k_1$ at the equilibrium $(k_1^*, k_2^*)$, we have
\[
\frac{\partial U^{(1)}}{\partial k_1} \bigg|_{(k_1,k_2)=(k_1^*,k_2^*)} = u'(c_1)(f'_1(k_1) + 1 - d) > 0. \quad (69)
\]

The partial derivative of $U^{(1)}$ with respect to $k_2$ is the first order condition in Eq. (32). Therefore, we have
\[
\frac{\partial U^{(1)}}{\partial k_2} \bigg|_{(k_1,k_2)=(k_1^*,k_2^*)} = 0. \quad (70)
\]

From Lemma 2, we know that there exist small values $\Delta k_1 > 0$ and $\Delta k_2 > 0$ such that $\frac{\partial U^{(0)}}{\partial k_1} \Delta k_1 + \frac{\partial U^{(0)}}{\partial k_2} \Delta k_2 > 0$ and $\frac{\partial U^{(1)}}{\partial k_1} \Delta k_1 + \frac{\partial U^{(1)}}{\partial k_2} \Delta k_2 > 0$. From Eqs. (69) and (70), we know that any marginal increase in $k_1$ would increase $U^{(1)}$. However, from Eqs. (64) and (67), marginal increases in capital $(\Delta k_1, \Delta k_2)$ must satisfy the following inequality to have $U^{(0)}$ increase:
\[
\frac{\Delta k_2}{\Delta k_1} > \frac{-u'(c_0) + \beta \delta u'(c_1)(f'_1(k_1) + 1 - d)}{\beta \delta (-u'(c_1) + \delta u'(c_2)(f'_2(k_2) + 1 - d))}. \quad (71)
\]

At the equilibrium where $(m_1, m_2) = (0, 0)$, from Eqs. (32) and (39) we can show that the inequality in Eq. (71) is equivalent to the following inequality:
\[
\frac{\Delta k_2}{\Delta k_1} > f_2'(k_1). \quad (72)
\]

Next, we need to show that if there exist small increases in $(m_1, m_2)$ that satisfy the inequality in Eq. (72), all intertemporal utilities increase. From Proposition 1,
we have
\[
\frac{dk_2}{dm_1} / \frac{dk_1}{dm_1} = k_2'(k_1),
\]
(73)
which is equivalent to
\[
\frac{dk_2}{dm_1} \Delta m_1 = \frac{dk_1}{dm_1} k_2'(k_1) \Delta m_1.
\]
(74)
From Proposition 2, we have
\[
\frac{dk_2}{dm_2} \Delta m_2 > k_2'(k_1) \frac{dk_1}{dm_2} \Delta m_2.
\]
(75)
From Eqs. (74) and (75), we have
\[
\frac{dk_2}{dm_1} \Delta m_1 + \frac{dk_2}{dm_2} \Delta m_2 > k_2'(k_1) \left( \frac{dk_1}{dm_1} \Delta m_1 + \frac{dk_1}{dm_2} \Delta m_2 \right),
\]
(76)
which is equivalent to the inequality in Eq. (72).

E. Proof of Proposition 4

We can prove it in the same way as in Proposition 3.

F. Proof of Proposition 5

Laibson (1997) introduced a steady-state model with quasi-hyperbolic discounting. In this section, we adopt Laibson’s steady-state model with the addition of optimism. In the steady state, we assume that consumers have homogeneous optimistic views, i.e., \( m_t = m_{t+1} \) for all \( t = 1, 2, \ldots, \infty \).

First, we introduce a partial equilibrium model, as first introduced in Laibson (1996) and which can be later extended to a steady state general equilibrium model. There are \( T \) periods. In each period, the consumer makes consumption-savings decisions. The consumer in period \( t \) chooses a consumption for period \( t \) as
\[
0 < c_t < W_t,
\]
then, period \( t + 1 \) consumer’s wealth is
\[
W_{t+1} = R(W_t - c_t).
\]
where $R$ is the gross real return on wealth, which is constant over time. The wealth $W_t$ represents the sum of present value of the stream of profits, labor and capital income.

The budget constraint in period $t$ is

$$c_t + k_{t+1} = R_t k_t + w_t + \pi_t.$$  \hspace{1cm} (77)

An increase in one unit of $c_t$ results in a decrease in one unit of $k_{t+1}$. The marginal benefit of postponing $\Delta$ units of consumption generates a stream of utility perturbations from the perspective of period $t$ consumer. At time $t$, the utility value of

$$\Delta u'(c_t)$$ \hspace{1cm} (78)

is lost. At time $t + 1$

$$\beta \delta \frac{\partial c_{t+1}}{\partial W_{t+1}} R \Delta u'(c_{t+1})$$ \hspace{1cm} (79)

units of utilities are gained. Note that $\frac{\partial c_{t+1}}{\partial W_{t+j}}$ is the marginal consumption rate at period $t + j$. At time $t + 2$, the utility gain is

$$\beta \delta^2 \frac{\partial c_{t+2}}{\partial W_{t+2}} \left( 1 - \frac{\partial c_{t+1}}{\partial W_{t+1}} \right) R^2 \Delta u'(c_{t+2}).$$ \hspace{1cm} (80)

At time $t + i$, the utility gain is

$$\beta \delta^i \frac{\partial c_{t+i}}{\partial W_{t+i}} \left[ \prod_{j=1}^{i-1} \left( 1 - \frac{\partial c_{t+j}}{\partial W_{t+j}} \right) \right] R^i \Delta u'(c_{t+i}).$$ \hspace{1cm} (81)

From Eqs. (78) and (81), we have

$$u'(c_t) = \beta \sum_{i=1}^{T-t} \delta^i \frac{\partial c_{t+i}}{\partial W_{t+i}} \left[ \prod_{j=1}^{i-1} \left( 1 - \frac{\partial c_{t+j}}{\partial W_{t+j}} \right) \right] R^i u'(c_{t+i}).$$ \hspace{1cm} (82)

The Euler equation for period $t + 1$ is

$$u'(c_{t+1}) = \beta \sum_{i=1}^{T-(t+1)} \delta^i \frac{\partial c_{t+1+i}}{\partial W_{t+1+i}} \left[ \prod_{j=1}^{i-1} \left( 1 - \frac{\partial c_{t+1+j}}{\partial W_{t+1+j}} \right) \right] R^i u'(c_{t+1+i}).$$ \hspace{1cm} (83)

From Eq. (82) and (83), we derive the following Euler equation on the unique equilibrium path:

$$u'(c_t) = R \delta u'(c_{t+1}) \left[ \beta \frac{\partial c_{t+1}}{\partial W_{t+1}} + \left( 1 - \frac{\partial c_{t+1}}{\partial W_{t+1}} \right) \right]$$ \hspace{1cm} (84)

Assume that consumption in each period can be characterized by the following
rule:
\[ c_t = \lambda_{T-t} W_t, \]
where the sequence \( \{\lambda_i\}_{i=0}^T \) is given by the recursion,
\[ \lambda_{i+1} = \frac{\lambda_i}{\delta R^{1-\rho} (\lambda_i (\beta - 1) + 1)^{1/\rho} + \lambda_i} \quad \text{and} \lambda_0 = 1. \quad (85) \]

As \( T \to \infty \), \( c_t \) converges to \( \lambda^* W \), where \( \lambda^* \) can be derived from Eq. (85):
\[ \lambda^* = 1 - (\delta R^{1-\rho})^{1/\rho} [\lambda^* (\beta - 1) + 1]^{1/\rho}. \quad (86) \]
With proportional consumption, the steady state condition is
\[ R(1 - \lambda^*) = \exp(g). \quad (87) \]
From Eqs. (86) and (87), we have
\[ \exp(\rho g) = \beta \delta R + (1 - \beta) \delta \exp(g). \quad (88) \]
where the equilibrium real rental rate of capital in the steady state is
\[ r^* = R - (1 - d) = \alpha \frac{Y}{K} (1 + m). \quad (89) \]
From Eqs. (88) and (89), we have
\[ \frac{K^*}{Y^*} = \frac{(1 + m)\alpha \beta \delta}{\exp(\rho g) - \delta \exp(g) (1 - \beta) - (1 - d) \beta \delta}. \]

References


